Journal of Statistical Physics, Vol. 79, Nos. 3/4, 1995

Escape Statistics for Systems Driven by Dichotomous Noise. II. The Imperfect Pitchfork Bifurcation as a Case Study

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Received June 3, 1994; final October 31, 1994

We use the general results for the escape probabilities and mean exit times obtained in an accompanying paper to analyze in detail a nonlinear system presenting an imperfect (subcritical) pitchfork bifurcation. We redraw the bifurcation diagram to show the effect of the noise. To avoid spurious results we introduce the concept of *extinction level* as the minimum possible value for the system, and discuss its effect on the bifurcation diagram.

KEY WORDS: Escape probabilities; mean first passage times; dichotomous noise, stochastic bifurcations; level of extinction.

1. INTRODUCTION

In an accompanying paper[1] (hereafter called I) we studied a one-dimensional stochastic nonlinear dynamical system

$$\dot{x}_t = F(x_t, \xi_t) \tag{1}$$

where F is a nonlinear function and ξ_t is a symmetric dichotomous noise which can take values $\pm\Delta$ with correlation time $\tau_c = 1/2\lambda$. Time between switches in the noise value, Δ or $-\Delta$, is governed by the distribution $\phi(t) = \lambda \exp(-\lambda t)$, and the average residence time in each of these states is $1/\lambda$.[2]

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0022-4715/95/0500-0683$07.50/0

In order to obtain the escape statistics for (1) we considered in I the quantities $f^{\pm}_{a,b}(t \mid x_0)\, dt$ defined as the probabilities of first reaching, in the time interval $(t, t+dt)$, the boundary a or b starting at $x_0 \in [a, b]$ and taking into account the initial value of the noise $\pm \Delta$. By deriving the differential equation satisfied by the Laplace transform $\tilde{f}$, we were able to get closed expressions for the escape probabilities and the conditional mean first passage times (MFPT). In particular, the equations for the exit probabilities are [the corresponding expressions for $P^{-}_{a,b}(x_0)$ are obtained switching a to b and $+$ to $-$]

$$P^{+}_{a}(x_0) = \frac{\lambda}{1+\lambda g^{+}(a)}\, g^{+}(x_0) \tag{2}$$

$$P^{+}_{b}(x_0) = 1 - P^{+}_{a}(x_0) \tag{3}$$

with

$$g^{+}(x) = \int_{x}^{b} dy\, \frac{1}{F_{+}(y)} \exp\left[\lambda \int_{a}^{y} ds \left(\frac{1}{F_{+}(s)} + \frac{1}{F_{-}(s)}\right)\right] \tag{4}$$

where $F_{\pm}(x) = F(x, \pm\Delta)$. For the conditional MFPT we have

$$T^{\pm}_{a,b}(x_0) = \frac{1}{P^{\pm}_{a,b}(x_0)} \left[\int_{b}^{x_0} dy\, V^{\pm}_{a,b}(y)\, e^{-M^{\pm}(y)} + C \int_{b}^{x_0} dy\, e^{-M^{\pm}(y)}\right] \tag{5}$$

where

$$M^{\pm}(x) = \int^{x} dy \left[\frac{F'_{\pm}(y)}{F_{\pm}(y)} - \lambda\left(\frac{1}{F_{+}(y)} + \frac{1}{F_{-}(y)}\right)\right] \tag{6}$$

$$V^{\pm}_{a,b}(x) = \int^{x} dy\, e^{M^{\pm}(y)} \left[\frac{2\lambda}{F_{+}(y)\, F_{-}(y)} - \left(\frac{1}{F_{+}(y)} + \frac{1}{F_{-}(y)}\right)\frac{\partial}{\partial y}\right] P^{\pm}_{a,b}(y) \tag{7}$$

and the constant C is fixed by the boundary conditions

$$\left.\frac{d}{dx}\right|_{x=b} [P^{-}_{a,b} T^{-}_{a,b}](x) = \frac{\lambda[P^{-}_{a,b} T^{-}_{a,b}](b) - P^{-}_{a,b}(b)}{F_{-}(b)} \tag{8}$$

$$\left.\frac{d}{dx}\right|_{x=a} [P^{+}_{a,b} T^{+}_{a,b}](x) = \frac{\lambda[P^{+}_{a,b} T^{+}_{a,b}](a) - P^{+}_{a,b}(a)}{F_{+}(a)} \tag{9}$$

Although the above expressions are general, to get additional insight we consider in this work a particular dynamical system of potential wide application: the imperfect (or subcritical) pitchfork bifurcation.[3,4] It is important to realize that any other one-dimensional model can be analyzed following the same guidelines exposed here.

The organization of the paper is as follows. Section 2 deals with a qualitative discussion of the asymptotic behavior of a one-dimensional general system driven by dichotomous noise. In Section 3 we study in detail the bifurcation diagram, escape probabilities, and MFPT of the chosen explicit example. We also discuss in this section the influence of an *extinction level*, i.e., a threshold $\varepsilon > 0$ below which the process is trapped, as happens, for instance, when the process is a chemical concentration $x = N/V$, for which a value $x < \varepsilon = O(1/N)$ is actually zero. We finally present our main conclusions in Section 4.

2. ASYMPTOTIC QUALITATIVE BEHAVIOR OF SYSTEMS DRIVEN BY DICHOTOMOUS NOISE

As is the case for the deterministic situation, it is very useful to know the general qualitative behavior of the system. For the stochastic case (1) the asymptotic evolution can be completely described considering the interval $[a, b]$ between two successive zeros of F_+ and F_-. Let us first consider the situation when the boundaries of the interval are not steady solutions,[5] i.e., they are zeros of only one of the forces.

If both boundaries are zeros of F_+ (Fig. 1a), the trajectories will escape with probability one in finite time through a, and will never return to the interior of the interval. If both are zeros of F_- (Fig. 1b), we have escape through b with the same conditions.

If $F_-(b) = F_+(a) = 0$ (Fig. 1c), the trajectories escape with probability one in finite time and never return to the interval, but now some trajectories will escape through a and some through b. This is a typical *bistability region*, and using (2), one can calculate the exit probability through either of the boundaries in terms of the initial conditions of the process x_t and the noise ξ_t.

If $F_+(b) = F_-(a) = 0$ (Fig. 1d), all trajectories remain confined within $[a, b]$. The interval is then an *invariant* set, and there are stationary, ergodic, Markov solutions $x(t)$ in it.[5] On the other hand, for any $\varepsilon > 0$, the probability of reaching $b - \varepsilon$ starting at $x_0 \in [a, b - \varepsilon]$ is 1, and so is the probability of reaching $a + \varepsilon$ starting at $x_0 \in [a + \varepsilon, b]$. This means that the trajectories in $[a, b]$ move in the whole interval and hence the stationary probability distribution has support $[a, b]$.

The situation in which a boundary is a steady solution of the stochastic system (Fig. 1e) is more delicate. The stability of the steady solution is governed by the Lyapunov exponent. As the noise is symmetric, it

[5] Throughout this paper we use the terms steady solution or steady state to indicate fixed points of the stochastic flow, i.e., a zero of both forces $F_\pm$.

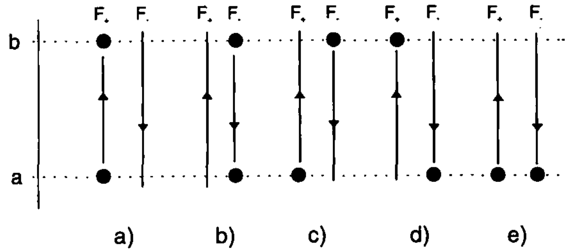

Fig. 1. Possible regions with stationary points of the flows $F_{\pm}(x)$. The large black points represent the zeros of the flows.

can be easily shown[6] that for the stochastic system, the Lyapunov exponent is the same as the one obtained for the same stationary solution of the deterministic counterpart $\dot{x}_t = F(x_t, 0)$.

For a *stable* steady point, i.e., if the Lyapunov exponent is negative, we have exponential convergence in a neighborhood of the stable point, but the radius of this neighborhood *depends on the realization of the noise.*[7] This is the main difference with respect to the deterministic case, and it may play a significant role in the stochastic behavior, since it is easy to imagine a situation such that, no matter how close to the steady solution the system starts, there exist realizations of the noise with an attraction neighborhood so small that the trajectory never reaches it, and finally moves away from the steady point.

If both forces are linearizable, i.e., with the same notation as I, $F_+(x) = x/\alpha + O(x^2)$ and $F_-(x) = -x/\beta + O(x^2)$, the dominant terms in the integrals in (4) are those corresponding to the linearization of the system. Consequently, the escape probability $P_b(x)$ from an interval $[0, b]$ behaves qualitatively as in the linear case, i.e., as calculated in I, we have escape with probability one if $\alpha \leqslant \beta$, whereas for $\alpha > \beta$ there is a nonzero probability, but less than one, of escaping through b, provided that $F_+(b) \neq 0$ (see the appendix for a more detailed discussion).

To finish this section, let us consider, for instance, that the lower boundary a is a steady point (Fig. 1e). Using the previous argument, we can prove that, when a is unstable, all trajectories escape through any upper nonstationary boundary, while if a is stable, there still is a nonvanishing probability of escape through b, provided that $F_+(b) \neq 0$. Therefore we can have bistability regions connected with stable steady solutions, where part of the trajectories tend to the steady state and the rest go away from it. This is a purely noise-induced effect that can make the asymptotic behavior of the stochastic system completely different from the deterministic situation.

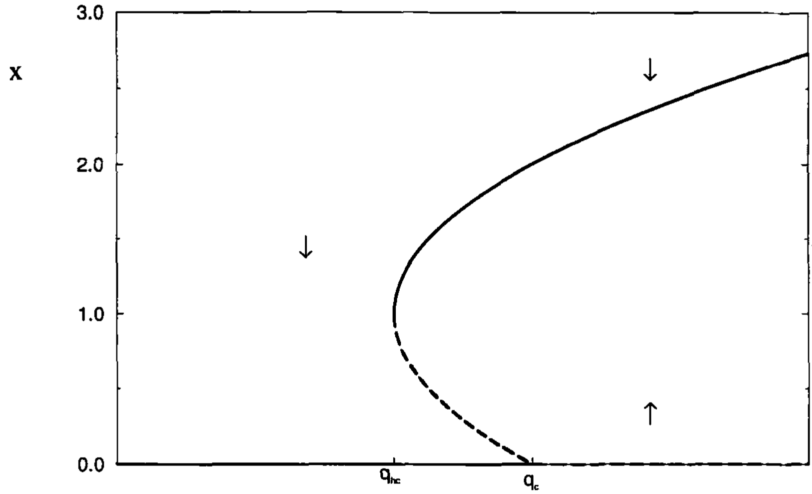

Fig. 2. Bifurcation diagram for the deterministic version of the system (10).

3. THE IMPERFECT PITCHFORK BIFURCATION

We consider the dynamical system

$$\dot{x} = -\tfrac{1}{2}x^3 + bx^2 + c(q_t - q_c)x \tag{10}$$

for $x \geqslant 0$, $c > 0$. The deterministic version ($q_t = q$) of this system shows an imperfect (subcritical) pitchfork bifurcation,[3,4] presenting a region of bistability, with hysteresis, for values of the bifurcation parameter q between two well-fixed values $q_{hc} = q_c - b^2/2c$ and q_c (see Fig. 2). We now assume that the bifurcation parameter is perturbed by a dichotomous noise around its mean value q, i.e., $q_t = q + \xi_t$, and analyze the changes in the bifurcation diagram.[6]

3.1. Stochastic Bifurcation Diagrams

Notice that $x = 0$ is the only steady state for the stochastic system. The Lyapunov exponent for this solution is equal to the deterministic eigenvalue $\lambda_{\text{det}} = c(q - q_c)$, i.e., $x = 0$ is stable when $q < q_c$ and unstable when $q > q_c$.

[6] A related system, the Stratonovich model obtained from (10) with $b = 0$, has been considered by Behn and Schiele (8) but focusing only on the MFPT problem.

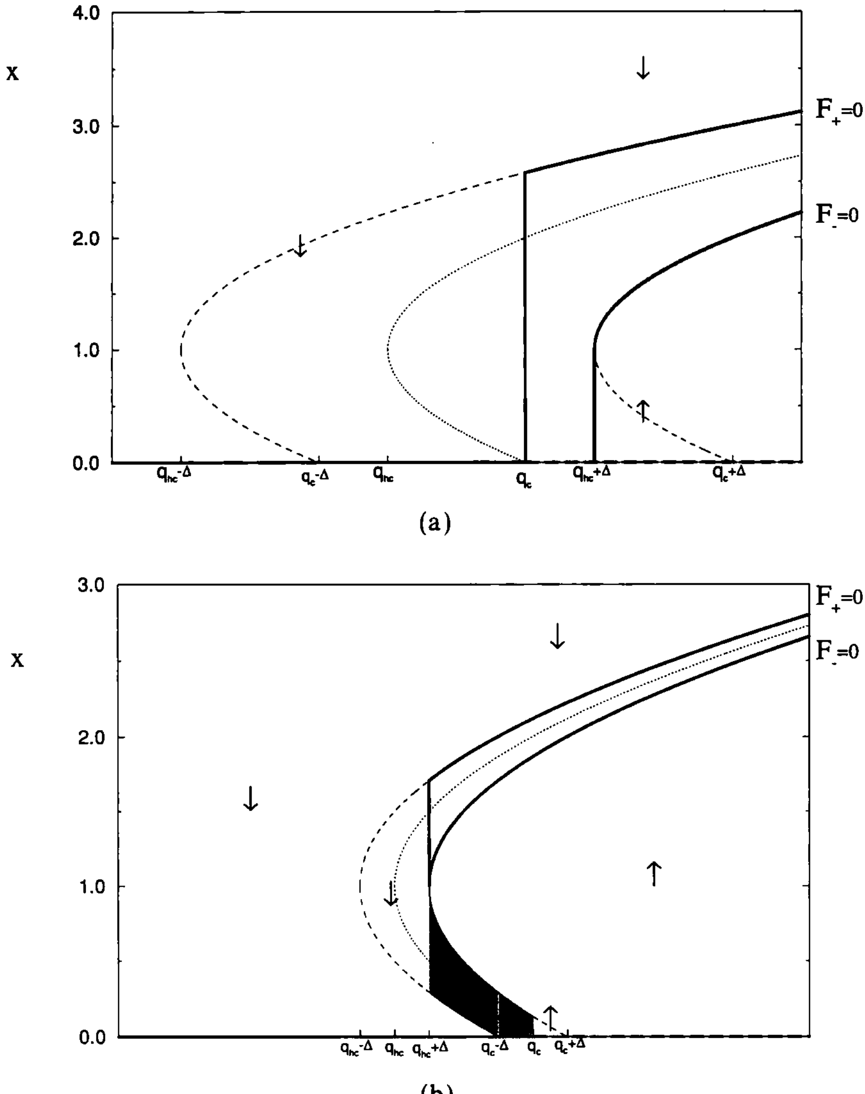

Fig. 3. Bifurcation diagrams of the stochastic version of the system (10) with $b=1$, $c=2.5$, $q_c=0.6$, and $q_{hc}=0.4$, depending on the intensity of the noise: (a) $\Delta=0.3>q_c-q_{hc}=b^2/2c$; (b) $\Delta=0.05<q_c-q_{hc}$ [the scale on the vertical axis is different from (a) to make the plot easier to see].

The nontrivial sets in which the forces $F_{\pm}$ point to opposite directions, and therefore may contain stationary Markov solutions, are those between the two parabolas

$$-\tfrac{1}{2}x^2 + bx + c(q - q_c \pm \Delta) = 0 \tag{11}$$

and we will use the notation

$$s_{\pm}(q) \equiv b + [b^2 + 2c(q - q_c \pm \Delta)]^{1/2} \tag{12}$$

$$r_{\pm}(q) \equiv b - [b^2 + 2c(q - q_c \pm \Delta)]^{1/2} \tag{13}$$

for the boundaries of these sets. For illustrative purposes two diagrams are depicted: the first one (Fig. 3a) for a value of $\Delta > q_c - q_{hc}$, large enough to mix the effects of the two deterministic bifurcation points q_c and q_{hc}. The second one (Fig. 3b), for the case $\Delta < (q_c - q_{hc})/2$, where the noise affects separately each deterministic bifurcation point. (The intermediate cases do not contain any qualitative aspect not present in the two diagrams selected.) In both diagrams the dotted parabola corresponds to the deterministic bifurcation diagram (included to make easier the comparison between the stochastic and the deterministic cases), and the dashed ones to the solutions of $F_+(x, q) = 0$ and $F_-(x, q) = 0$. There $s_{\pm}$ and $r_{\pm}$ are the upper and lower branches of the external and internal parabolas (zeros of F_+ and F_-).

For a fixed value of q, and in the limit $t \to \infty$, the trajectories will either be confined between the thick lines or tend to $x = 0$. Projection of the intervals between these thick lines onto the y axis gives the exact support region of the corresponding stationary probability distribution. The arrows indicate the behavior depending on the initial conditions, and the filled regions, if any, denote bistability.

The asymptotic behavior in each diagram, depending on q (mean value of the stochastic control parameter q_t), is as follows.

3.1.1. Large $\Delta(\Delta > q_c - q_{hc})$. We may distinguish several situations (Fig. 3a).

(a) $q < q_{hc} - \Delta$. All trajectories approach the only stationary stable solution $x = 0$.

(b) $q_{hc} - \Delta < q < q_c - \Delta$. Again all trajectories approach 0 with probability one. The trajectories slow down when they enter $[r_+(q), q_+(q)]$, since there F_+ and F_- point to opposite directions [notice that both boundaries are zeros of F_+ and all trajectories exit through $r_+(q)$].

(c) $q_c - \Delta < q < q_c$. Every trajectory will enter the interval $[0, s_+(q)]$. On the other hand, $x = 0$ is stable, i.e., we have exponential convergence toward 0 in a neighborhood that depends on the realization of the noise. A simple argument based on escape probabilities shows that, again, the only stationary solution is $x=0$. Notice that according to the results in I, for any $\varepsilon<s_+(q)$, the probability that a trajectory in $[\varepsilon, s_+(q))]$ exists through ε is one. Then, all trajectories will reach, sooner or later, an arbitrarily small neighborhood of $x=0$, being trapped by this solution.

(d) $q_c<q<q_{hc}+\Delta$. Since no trajectory can leave $[0, s_+(q)]$, there are stationary Markov solutions in this interval. Now, $x=0$ is unstable and no trajectory is trapped. With the same argument as stated for the example of Fig. 1d in the previous section, we conclude that the exact support of the stationary distribution is $[0, s_+(q)]$.

(e) $q_{hc}+\Delta<q<q_c+\Delta$. Trajectories starting in $[0, r_-(q)]$ leave this interval through r_-. Then, all trajectories with $x_0>0$ will reach the interval $[s_-(q), s_+(q)]$. A similar argument as in the previous case shows that this interval is the support of the distribution of stationary Markov solutions.

(f) $q_c+\Delta<q$. Stationary nonzero solutions lie in $[s_-(q), s_+(q)]$ and their distribution has support on the whole interval.

3.1.2. Small $\Delta (\Delta<(q_c-q_{hc})/2)$. The asymptotic behavior is the same as in the case of Section 3.1.1, except if $q_{hc}+\Delta<q<q_c$ (Fig. 3b). For this q interval we have a bistability region (filled in the figure). The strict domain of attraction of stationary solutions in $[s_-(q), s_+(q)]$ is $x\geqslant r_-(q)$. We get a strict domain of attraction $[0, r_+(q)]$ for $x=0$ when $q_{hc}+\Delta<q<q_c-\Delta$, while if $q_c-\Delta<q<q_c$, trajectories starting at $x\in[0, r_-(q)]$ can approach the upper stationary solutions or 0. The probability that such a trajectory approaches 0 is equal to 1—the probability of exit through $r_-(q)$.

3.2. Escape Statistics

In order to draw the stochastic bifurcation diagrams we made use only of the stability properties of the steady states, via their Lyapunov exponents, and of the value of escape probabilities through certain boundaries. In fact we only need to know if these probabilities are zero, one, or less than one.

The most important information about the transient regime is contained in the probability distributions $f^{\pm}_{a,b}$. The exact knowledge of their zeroth-order moments, i.e., the escape probabilities, gives us the fraction of trajectories that escape from a region through each boundary, and the higher-order moments of the distributions can be used to characterize fully the time in which a trajectory reaches its final stationary regime.

Intervals where both forces $F_{\pm}$ point to different directions but do not contain stationary solutions are also of special interest. Every trajectory starting in such an interval will escape sooner or later, but they can be considerably delayed and the time needed to escape from these regions can be large enough to make them appear as metastable states.

Next we will show the escape probability and mean escape time for different situations of this type located in the bistability region of the bifurcation diagram in Fig. 3b. We have found problems of convergence in the numerical calculation of the integrals involved in the expressions for the escape probabilities and mean escape times. To overcome these problems we use a trick, explained in the appendix, that connects the escape statistics of the linearized system from a small interval $[0, a]$ with the escape statistics of the nonlinear system from the whole interval $[0, b]$.

Figures 4a and 4b correspond to a q value such that $q_{hc} + \Delta < q < q_c - \Delta$, and represent the escape probability from the interval $[r_+, r_-]$ through the upper boundary and the corresponding conditional exit time, respectively, for some values of λ (half of the inverse of the noise correlation time).

The transition from constant noise $\lambda \to 0$ to the deterministic system[7] $\lambda \to \infty$ can be seen in Fig. 4. For constant noise the exit through both boundaries is equally probable, independent of x_0 (a simple consequence of the averaging over the two possible initial conditions of the noise). For $\lambda \to \infty$ the probability is one or zero, depending on the initial condition.

In Fig. 4c we depict the mean first passage time through any boundary, i.e., the escape event no matter which of the boundaries is first crossed, from the same interval. For large λ values, as the initial condition of the trajectory is determinant for the escape event, we find a curve that reproduces the intuitive expectations about the escape time: it has a maximum in the middle of the interval. On the other hand, for smaller λ the behavior is completely different and the escape time has a minimum. The transition between having a maximum or a minimum occurs close to $\lambda = 0.1$.

Figures 5a and 5b correspond to $q_c - \Delta < q < q_c$ and represent the probability of escape from $[0, r_-]$ through the upper boundary r_- and the corresponding conditional mean first passage time. For small λ the escape probability approaches $(\alpha + \beta)/2\alpha$ (0.55 for the values considered in

[7] Recall that the dichotomous noise becomes a white noise when simultaneously λ and Δ go to infinity. If the correlation time is set equal to zero keeping Δ finite, one recovers the deterministic equation.

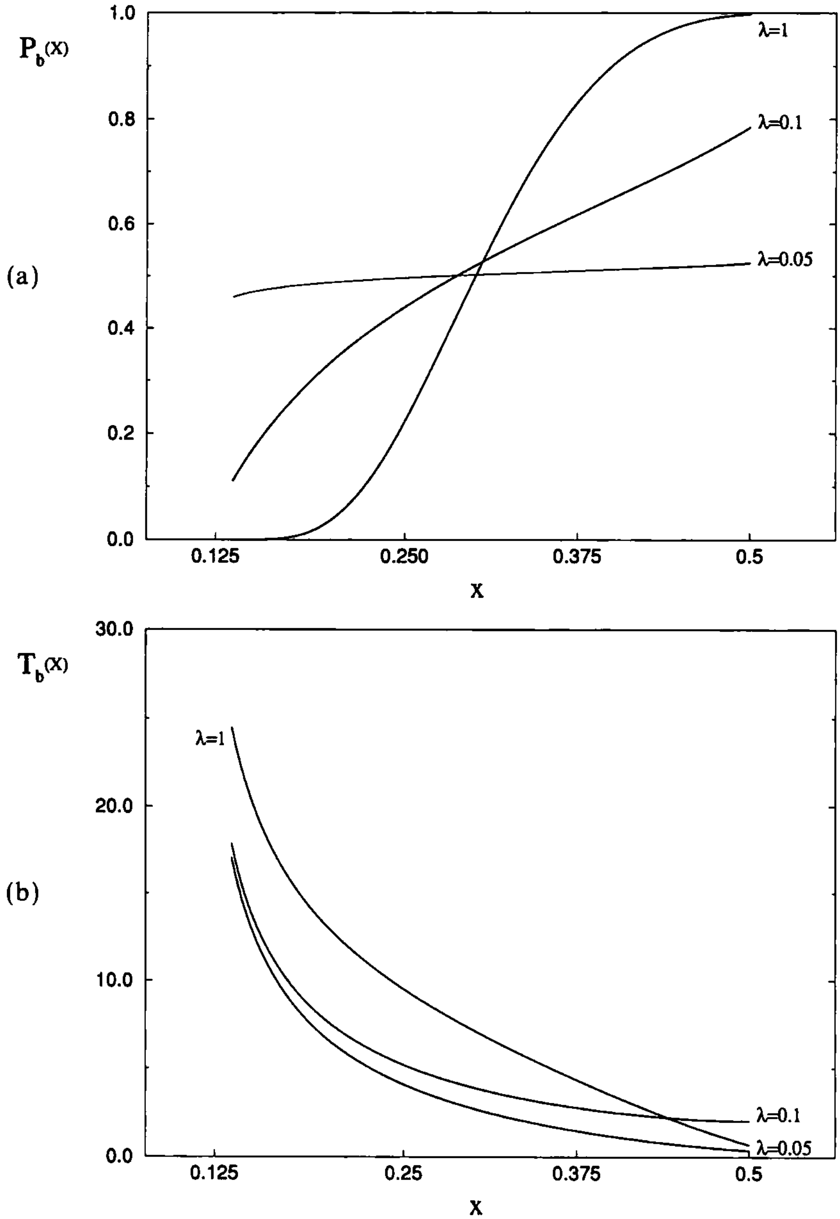

Fig. 4. (a) Escape probabilities, (b) conditional mean escape time, and (c) usual MFPT for $q = 0.5$ ($q_{hc} + \Delta < q < q_c - \Delta$) and various values of λ. The other parameters are the same as in Fig. 3b. The interval here is $[r_+, r_-]$.

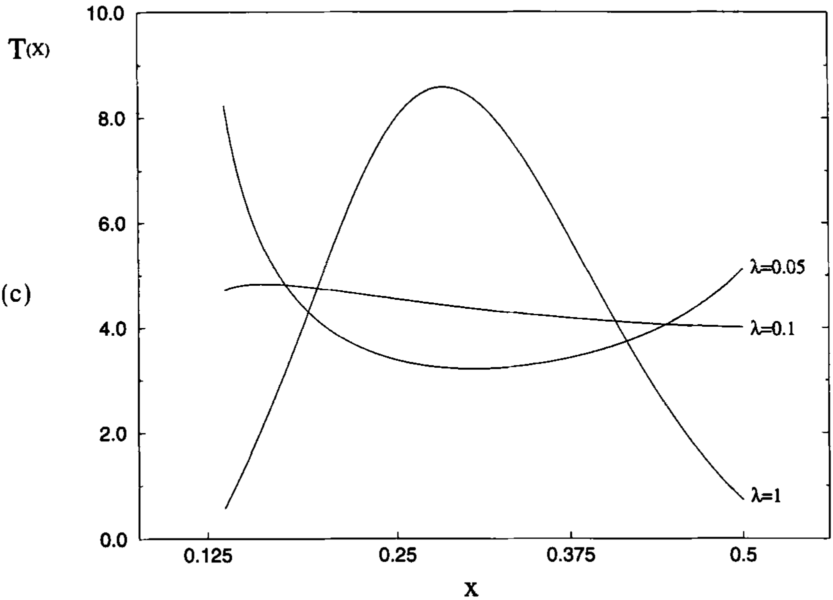

Fig. 4. *(Continued)*

Fig. 5a) as in the linear case studied in I, independent of the initial condition of the trajectory. Observe that this value does not equal 1/2, the intuitively expected result for $\lambda = 0$ (constant noise). The explanation of this discrepancy lies in the fact that the limits $\lambda \to 0$ and $a \to 0$ do not commute, as was explained in I. Notice that as λ decreases, this escape time increases in a very similar way to the linear case, and its numerical values are considerably higher than in other events of the system.

3.3. Extinction Level

The use of differential equations to describe the evolution of real systems is only a convenient continuous approximation of a process that is actually discrete. This is particularly clear in the study of chemical reactions or in biological applications, where x represents the *concentration*, i.e., the number of particles/molecules/individuals divided by a characteristic size/volume of the system. In order to avoid spurious results with no relation to the system one tries to model, this fact should be kept in mind when analyzing the asymptotic, $t \to \infty$, behavior, mainly if the system is stochastic and its evolution may well be contrary to our deterministic intuition.

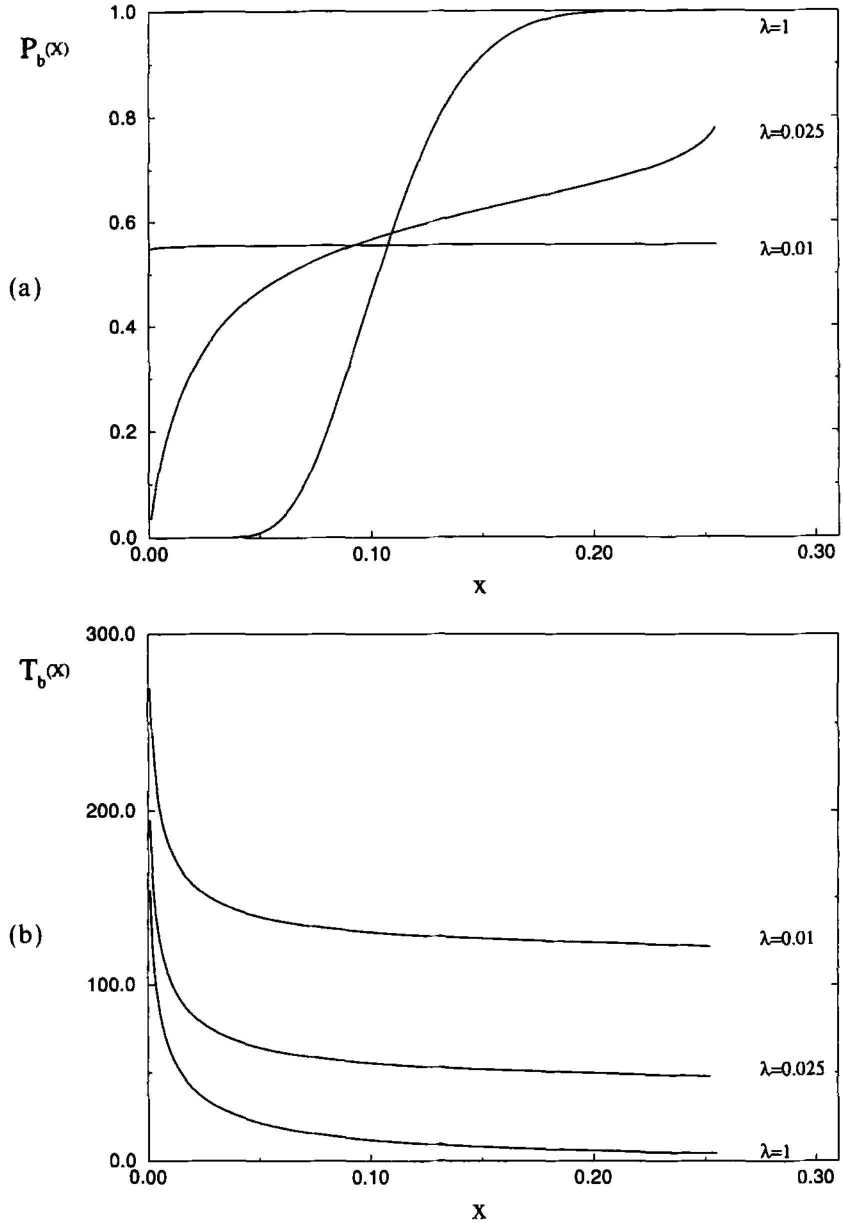

Fig. 5. (a) Escape probabilities and (b) conditional mean escape time for $q = 0.56$ ($q_{c-\Delta} < q < q_c$) and various values of λ. The other parameters are the same as in Fig. 3b. The interval here is $[0, r_-]$.

As an example of the above arguments, the deterministic version of Eq. (10) appears in the study of hypercycles of self-replicative molecules,[9] where x represents the relative concentration of certain molecules in the hypercyclic organization, i.e., those molecules with a catalytic relationship that improve their self-reproductive rates. To avoid the problem of considering too low concentrations for which hypercyclic growth seems to be unrealistic, a *level of extinction*, i.e., a fixed minimal concentration $x = \varepsilon$ as representative of the extinction of the species instead of 0 was introduced in ref. 10.

In our case of dichotomous perturbations we can use the general exact results of I to find the behavior of the system including an extinction level by considering the escape statistics from an interval $[\varepsilon, b]$. Exit through ε obviously means extinction, whereas b can be an appropriate threshold to ensure that the system reaches a stationary state far from the extinction level. Several effects can be induced, and we focus our attention on those appreciable enough even for a very small extinction level ε.

First, let us analyze how the inclusion of this effective extinction level modifies the bifurcation diagram of Fig. 3a corresponding to a large value of the noise. It is clear that regions where both forces point to the same direction are not affected. Now consider q between q_c and $q_{hc} + \Delta$. The support of the stationary distribution is the whole interval $[0, r_+]$, but, since the system is ergodic, every trajectory moves around the entire interval and hence they will reach, sooner or later, the extinction level. Then, for this

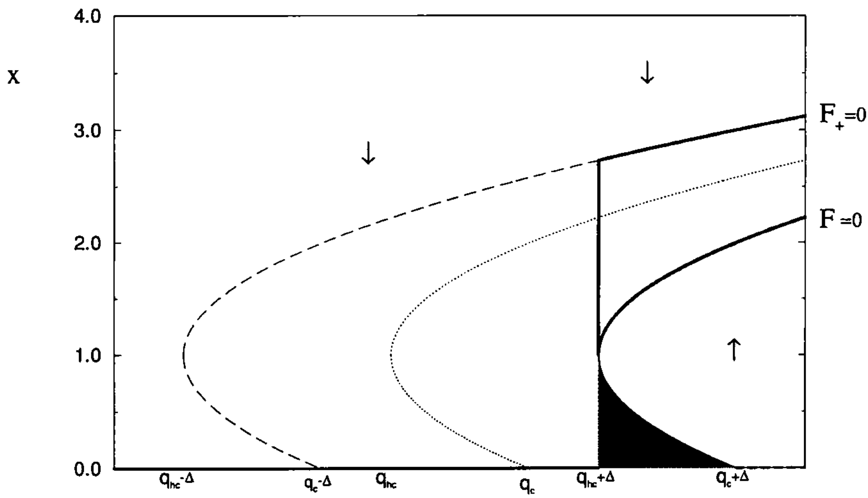

Fig. 6. The same bifurcation diagram as Fig. 3a, modified by the inclusion of an extinction level. The shaded region is the bistability area created by the presence of the extinction level.

value of q there are no stationary solutions different from the total extinction. For q between $q_{hc}+\Delta$ and $q_c+\Delta$ the stationary state was an invariant density defined on the interval $[r_+, r_-]$ and every trajectory moved inside. However, when considering an extinction level, these regions become bistability regions where, depending on the initial condition, trajectories have a nonzero probability of reaching the extinction level. The resulting bifurcation diagram is depicted in Fig. 6. Note that these are drastic modifications of the asymptotic behavior of the system that occur no matter how small ε is.

The second relevant effect is related to the noncommuting limits $\lambda \to 0$ and $a \to 0$ discussed previously. Figure 7 shows that, for the probability $P_b^-(x_0)$ of escaping through the upper boundary, when the noise is initially $-\Delta$, the limit $\lambda \to 0$ abruptly changes when including an extinction level. The reason is that $\lambda \to 0$ is the constant noise limit and the process x_t will reach the extinction level before any switch of the noise. This does not happen whithout an extinction level, since then the process does not reach 0 in finite time and therefore, for λ small but nonzero, there is always a nonvanishing probability of switching and moving upward to the exit. Note that here the effect is also independent of the value of ε: no matter how small ε is, in the limit $\lambda \to 0$, $P_b^-(x_0)$ vanishes.

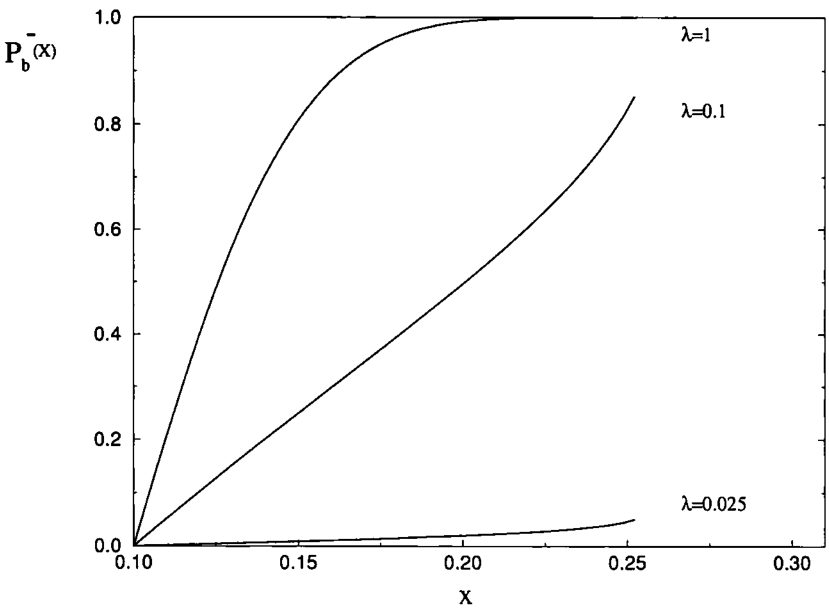

Fig. 7. Effect of the extinction level on the probabilites of Fig. 5a.

4. CONCLUSIONS

We have analyzed in detail the behavior of a dynamical system that under deterministic conditions presents an imperfect pitchfork bifurcation when the control parameter is perturbed by a dichotomous noise. Using the general results contained in I on exit probabilities and the stability of the fixed points of the stochastic flow, we can draw the new stochastic bifurcation diagram as a function of the mean value of the fluctuating control parameter and identify the regions where the deterministic and the stochastic behavior are different. In particular, it is interesting to note that when the noise can take values in a range larger than the deterministic bistability region, i.e., $\Delta > q_c - q_{hc}$, the bistability area disappears completely, whereas if Δ is smaller, the bistability region exists, but is reduced with respect to the deterministic one. The introduction of an arbitrary level of extinction, i.e., a minimum value below which the system is absorbed, also drastically modifies the bifurcation diagram, by creating a new bistability area, not present in the deterministic nor in the large-range stochastic situations. As a second effect of the level of extinction $P_b^-(x_0)$, the escape probability through the upper boundary of an interval when the initial condition of the noise is $-\Delta$, drops to zero when the correlation time of the noise increases.

We would like to emphasize that, as already indicated in I, the results obtained in this work, together with the standard calculation of the stationary probability distributions with support in the invariant sets,[2] lead to a complete description of the evolution of the stochastic system.

As a final remark, we conjecture that qualitatively similar results should be obtained when considering more general bounded noises, such as, for example, the diffusion stochastic process considered in ref. 10. Therefore, the dichotomous noise could be taken as a useful guide in the analysis of more realistic models and/or noises.

APPENDIX

The integrals involved in the calculation of the exit probabilities and mean escape times considered in the paper are convergent, but some of them are improper at $a = 0$, and their numerical solution is a difficult task. In this appendix we propose a method based on probabilistic arguments to solve the problem (see I for notation).

Let us decompose the interval $[0, b]$ into two disjoint intervals: a small one around 0, $[0, a]$, and the rest, $[a, b]$. A trajectory starting at $x_0 > a$ can cross b in two ways: either the trajectory crosses b without crossing a or it crosses b after crossing a. Taking into account that a is always crossed with a noise value $+\Delta$ if upward and $-\Delta$ if downward, the last assertion implies for the probability densities of first reaching a or b the following relation:

$$f_b^{+,[0,b]}(t\,|\,x_0) = f_b^{+,[a,b]}(t\,|\,x_0) + \iiint_{t_1+t_2+t_3=t} dt_1\,dt_2\,dt_3\, f_a^{+,[a,b]}(t_1\,|\,x_0) \times f_a^{-,[0,a]}(t_2\,|\,a)\, f_b^{+,[0,b]}(t_3\,|\,a)$$

where we have indicated as superscripts the intervals of escape. Applying the Laplace transform, we get an algebraic relation

$$\tilde{f}_b^{+,[0,b]}(s\,|\,x_0) = \tilde{f}_b^{+,[a,b]}(s\,|\,x_0) + \tilde{f}_a^{+,[a,b]}(s\,|\,x_0)\, \tilde{f}_a^{-,[0,a]}(s\,|\,a)\, \tilde{f}_b^{+,[0,b]}(s\,|\,a)$$

The idea is to set a very close to zero in order to use the results obtained in I for the linearization of the system. The moments of the escape distributions corresponding to the interval $[a, b]$ can be calculated with the usual integration algorithms. The only remaining unknown quantity in the last expression is $\tilde{f}_b^{+,[0,b]}(s\,|\,a)$, but it can be obtained by setting $x = a$ in the same equation,

$$\tilde{f}_b^{+,[0,b]}(s\,|\,a) = \frac{\tilde{f}_b^{+,[a,b]}(s\,|\,a)}{1 - \tilde{f}_a^{+,[a,b]}(s\,|\,a)\, \tilde{f}_a^{-,[0,a]}(s\,|\,a)}$$

The same procedure can be used with $\tilde{f}_b^{-,[0,b]}(s\,|\,x_0)$. After some calculations we obtain the final expression for the escape probabilities and the mean escape times:

$$P_b^{[0,b]}(x_0) = P_b^{[a,b]}(x_0) + P_a^{[a,b]}(x_0)\, \frac{P_b^{+,[a,b]}(a)\, P_a^{-,[0,a]}(a)}{1 - P_a^{-,[0,a]}(a)\, P_a^{+,[a,b]}(a)}$$

$$T_b^{[0,b]}(x_0) = \frac{1}{P_b^{[0,b]}(x_0)} \,[P_b^{[a,b]}(x_0)\, T_b^{[a,b]}(x_0) + [T_a^{[a,b]}(x_0) + T_a^{-,[0,a]}(a) + T_b^{+,[0,b]}(a)] \times P_a^{[a,b]}(x_0)\, P_a^{-,[0,a]}(a)\, P_b^{+,[0,b]}(a)]$$

where

$$P_b^{+,[0,b]}(a) = \frac{P_b^{+,[a,b]}(a)}{1 - P_a^{-,[0,a]}(a)\, P_a^{+,[a,b]}(a)}$$

$$T_b^{+,[0,b]}(a) = T_b^{+,[a,bg]}(a) + [T_a^{+,[a,b]}(a) + T_a^{-,[0,a]}(a)] \times \frac{P_a^{-,[0,a]}(a)\, P_a^{+,[a,b]}(a)}{1 - P_a^{-,[0,a]}(a)\, P_a^{+,[a,b]}(a)}$$

As we pointed out before, the stability of $x = 0$ depends on the sign of the linear terms in the forces $F_\pm$ and their relative strength. Let us suppose again that these terms are of the form x/α, $-x/\beta$, and that we have an upper boundary b such that the interval $[0, b]$ does not contain any other zero of $F_\pm$. When $\alpha \leqslant \beta$ we have $P_a^{-,[0,a]} = 1$, and therefore

$$P_b^{[0,b]}(x_0) = 1$$

On the contrary, when $\alpha > \beta$, $0 < P_b^{[0,b]}(x_0) < 1$, i.e., there is a nonzero probability, but less than one, of escaping through b.

ACKNOWLEDGMENTS

This work has been supported by the Dirección General de Investigación Científica y Técnica (DGICYT), Spain, project PB91-0222.

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